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Equations of motion in a non-integer-dimensional space

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Abstract

Equations of motion are derived for a fractional dimensional system of n-spatial coordinates to be used as an effective description of anisotropic and confined systems. An existing measure theoretic approach is extended to multiple variables and different degrees of confinement in orthogonal directions and comparisons are made with the analytic continuation of Gaussian integrals. This is applied to the variational principle, and equations of motion for a field described by a Lagrange density are found. A specific example is looked at in Schrödinger wave mechanics, particularly in three-coordinate systems.

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1. Introduction

Fractional dimensional space has successfully been used as an effective physical description of confinement in low-dimensional systems. First applied by He [1–3], this approach replaces the real confining structure with an effective space, where the measure of its anisotropy or confinement is given by the non-integer dimension.

There are many approaches in use to describe fractional dimensions. These include fractal geometry [4], fractional calculus (a generalization of integration and differentiation to fractional order) [5] and the analytic continuation of the dimension in Gaussian integrals [6–10]. The latter is often used in quantum field theory as a regularization parameter in Feynmann diagrams [9, 10], and has also been used for investigations into Fermi and Luttinger liquids [11, 12].

Many of the investigations into low-dimensional semiconductor structures have used a mathematical basis introduced by Stillinger [6] in which he described integration on a space of D dimensions and provided a generalization of the Laplace operator on this space. Recent

progress includes the description of a single-coordinate momentum operator in this fractional dimensional space based on generalized Wigner commutation relations [13, 14] presenting a possible realization of parastatistics [15].

The formalism from [6] has been applied to problems such as excitons [3, 16–22], magnetoexciton [23], impurities [24], polarons [25] and superconductivity [26], often successfully mirroring computational results in specific cases. Nevertheless, all of these applications have been based on either one- or two-coordinate Laplacian and not three-coordinate Laplacian due to the increasing complexity of Stillinger's method as the number of variables increases. This is a problem in that the applications should intrinsically be in three coordinates and the number of coordinates has a great effect on the solution of equations like in the wave equation [27]. For example, it was noted in [16] that in the exciton problem only the two-coordinate model is really good for the description of s-states because it is highly symmetric. For a full description of the p-states or higher orbital angular momentum a full three-variable problem is desirable.

A question which the formalism of Stillinger brings up is how does the dimension arrange itself to act on the coordinates. With two orthogonal coordinates it fixes itself to a single variable only while leaving the other free. We believe that this is generally the case for n orthogonal coordinates as well. With this in mind is it possible to distribute the dimension D between the coordinates by attaching different dimensions to individual orthogonal coordinates? Physically this seems to be a useful idea to describe confinement in low-dimensional systems which can have different degrees of confinement in different orthogonal directions. So for example it may be more useful for a description of dots which are not in general spherical. If we have a system which is 1.6 dimensional, then different descriptions of this could be in two coordinates as $1 + 0.6$ dimensional and in three coordinates as $1 + 0.2 + 0.4$ dimensional, if the dimension adds linearly.

This paper develops a formalism in n spatial coordinates to address these questions. In the following section we briefly summarize the commonly used approach first proposed by Stillinger [6]. Several key results are identified which are often referred to throughout the paper. In section 3 we look at a different formalism in measure theory which allows us to construct the same single-variable integration as in Stillinger's method [6]. We use this in section 3.2 as a basis to develop a multiple-variable approach where each orthogonal coordinate has its own dimension. We then show that this not only includes Stillinger's integration method within it, but is also a significant generalization and appears more accessible. In section 4 we use a variational method to derive the equations of motion in a non-integer-dimensional space, including Stillinger's non-integer-dimensional Laplacian, in n coordinates. The approach can be used to find the equation of motion for any system if we have an appropriate Lagrangian density. Section 5 applies the approach to the Schrödinger equation, and also presents full analytic solutions for a three-variable system. Comparisons with the equations derived here and Stillinger's approach are made before finally concluding.

2. Stillinger's framework for integration on non-integer-dimensional spaces

To begin, we re-examine the main points of Stillinger's formalism [6] for integrating on a D-dimensional space. We consider n points $\mathbf{x}_i$ in a topological space with distances between points given by a metric $r_{ij} = r(\mathbf{x}_i, \mathbf{x}_j)$. Integration is then defined by the variation of a point $\mathbf{x}_0$ over all spaces where the volume element is the intersection of all the D-dimensional spherical shells drawn between $\mathbf{x}_0$ and $\mathbf{x}_i$ for all $i = 1, \ldots, n$. The volume element is written as $W_n(\mathbf{x}_1 \cdots \mathbf{x}_n | r_{01} \cdots r_{0n})$ and is a function of the $\frac{1}{2}n(n+1)$ distances $\{r_{ij} : 0 \leqslant i < j \leqslant n\}$.

Integration on a D-dimensional space is then defined as

$$\int \mathrm{d}\mathbf{x}_0 h(r_{01}\cdots r_{0n}) := \int_0^\infty \mathrm{d}r_{01}\cdots\int_0^\infty \mathrm{d}r_{0n} W_n(\mathbf{x}_1\cdots\mathbf{x}_n|r_{01}\cdots r_{0n})h(r_{01}\cdots r_{0n}) = f(\{r_{ij} : 1 \leqslant i < j \leqslant n\}). \tag{2.1}$$

To find the weights the integration is chosen to satisfy the following integral equation, where the dimension explicitly enters the formalism,

$$\int \mathrm{d}\mathbf{x}_0 \exp\left(-\sum_{j=1}^n \beta_j r_{0j}^2\right) = \left(\frac{\pi}{\tau}\right)^{D/2} \exp\left(-\frac{1}{\tau}\sum_{j<k=1}^n \beta_j\beta_k r_{jk}^2\right) \tag{2.2}$$

and the $\{\beta_j : 1 \leqslant j \leqslant n\}$ with real-valued constants and $\tau = \sum_{j=1}^n \beta_j$. From a repeated application of equation (2.2), Stillinger showed that the weights satisfy the following relation:

$$\int_0^\infty \mathrm{d}r_n W_n(\mathbf{x}_1\cdots\mathbf{x}_n|r_{01}\cdots r_{0n}) = W_{n-1}(\mathbf{x}_1\cdots\mathbf{x}_{n-1}|r_{01}\cdots r_{0(n-1)}). \tag{2.3}$$

While integration over a single variable was shown to be

$$\int \mathrm{d}\mathbf{x}_0\, f(r(\mathbf{x}_0,\mathbf{x}_1)) = \int_0^\infty \mathrm{d}r\; W_1(r) f(r) \tag{2.4}$$

with the weight given by

$$W_1(r) = \sigma(D) r^{D-1} \tag{2.5}$$

where

$$\sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{2.6}$$

From this Stillinger went on to derive a single-variable Laplacian in D dimensions,

$$\nabla^2\psi(r) = \psi''(r) + \frac{D-1}{r}\psi'(r). \tag{2.7}$$

In the case of two points, $\mathbf{x}_1$ and $\mathbf{x}_2$, integration can be written as

$$\int \mathrm{d}\mathbf{x}_0\, f(r_{01}, r_{02}) = \int \mathrm{d}p \int \mathrm{d}l\; J(p,l) W_2(p,l) f(p,l) = \int \mathrm{d}p \int \mathrm{d}l\, \sigma(D-1) l^{D-2} f(p,l) \tag{2.8}$$

which is expressed in terms of two variables p and l, which represent the line between $(\mathbf{x}_0, \mathbf{x}_1)$ and $(\mathbf{x}_1, \mathbf{x}_2)$ and its orthogonal complement respectively. The Jacobian, $J(p,\ l)$, accounts for the change of variable in equation (2.1) to the orthogonal pair. The two-variable integration weight led Stillinger to obtain an expression for a two-coordinate Laplacian as

$$\nabla^2\psi(p,l) = \left[\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial l^2} + \frac{D-2}{l}\frac{\partial}{\partial l}\right]\psi(p,l). \tag{2.9}$$

We see that the new term introduced for the fractional dimensional space acts only on the l component. Equation (2.9) has often provided the starting point for many of the subsequent investigations [3, 16–23], although it has often been stated that a three-variable form would be useful. In principle it is possible to use Stillinger's approach to do this but it appears awkward to manipulate and check results, due to the size of the expression of the integration weight W_3. This makes it difficult to progress in this method, hence in the following sections we develop and extend the method using a different approach.

3. Measure theory

A different approach, developed by Svozil [7] within the framework of measure theory [28], can be taken to arrive at the single-variable integration in equation (2.4). The attraction of this approach is the rich mathematical background of the subject. The first part of this section considers the development of the single-variable approach [7], which, in the second part, is then used to construct a multiple-variable approach through the introduction of product measures.

We begin by considering a measure space [28] (X, M, μ, d) which consists of a set X, and a collection of Borel subsets M and a metric d. μ measures X, if and only if $\mu : M \to \overline{\mathbb{R}} \cap \{t : 0 \leqslant t \leqslant \infty\}$ where $\overline{\mathbb{R}} = \mathbb{R} + \{-\infty\} + \{\infty\}$ and when $B \subset M$ and B is a countable sequence of sets

$$\mu(\cup B) \leqslant \sum_{B} \mu(B)$$

with equality if B are disjoint Borel sets.

We take the measure μ as the Hausdorff measure, over a set $E \subset M$,

$$\mu_{\mathcal{H}}(E, D) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \alpha(E_i, D)[d(E_i)]^D : E \subset \bigcup_i E_i, d(E_i) < \delta, \forall i \right\} \tag{3.1}$$

with D a positive real number and $\alpha(E_i, D)$ is a geometrical factor. The Hausdorff dimension is then given by

$$D = \dim_{\mathcal{H}}(E) = \inf\{s : \mu_{\mathcal{H}}(E) = 0\} = \sup\{s : \mu_{\mathcal{H}}(E) = \infty\}.$$

When D is an integer n, the Hausdorff measure becomes the n-dimensional Lebesgue measure and the counting measure when n is zero [4, 28]. Measures other than the Hausdorff measure could of course be considered and several are discussed in [28].

3.1. Single-variable integration

Following Svozil [7] we take $X \subset \mathbb{R}^n$ with n being a positive integer and elements $x, y, z, \ldots \in X$ are denoted by n-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$. Further conditions imposed on X are that it is *closed*, *unbounded* and *regular* so that D is the same or unique over all X with respect to the measure. Integration over a continuous function $f(x)$ may then be defined as the limit of an infinitesimal covering diameter where $\{E_i\}$ is a disjointed covering and $x_i \in E_i$ giving

$$\int_X F(x)\, \mathrm{d}\mu_{\mathcal{H}}(x) := \lim_{d(E_i) \to 0} \sum_{E_i} f(x_i)\alpha(E_i, D)[d(E_i)]^D. \tag{3.2}$$

For a spherically symmetric system we parametrize X by polar coordinates with $r = d(x, 0)$ and angles Ω. $E_{r,\Omega}$ can be thought of as a spherically symmetric covering around a centre at the origin. If our function $f(x)$ is symmetric with respect to some centre $x_0 \in X$, i.e. $f(x)$ is a constant $\forall\, x : d(x, x_0) = r$, then we can transform our space to shift the centre of symmetry to the origin. With this parametrization

$$\mathrm{d}\mu_{\mathcal{H}}(r, \Omega) = \lim_{d(E_{r,\Omega}) \to 0} \alpha(E_{r,\Omega}, D)[d(E_{r,\Omega})]^D = \mathrm{d}\Omega_{D-1} r^{D-1}\, \mathrm{d}r. \tag{3.3}$$

where $\mathrm{d}\Omega_{d-1}$ is defined by its integral being the volume of a D-dimensional sphere of unit radius. Explicitly,

$$\int \mathrm{d}\Omega_{d-1} = \frac{2\pi^d/2}{\Gamma(d/2)}. \tag{3.4}$$

Therefore for symmetric $f(x)$ we obtain for the D-dimensional integral

$$\int_X f(r)\,\mathrm{d}\mu_{\mathcal{H}} = \int_0^\infty f(r)\,\mathrm{d}\Omega_{D-1} r^{D-1}\,\mathrm{d}r$$
$$= \sigma(D)\int_0^\infty f(r) r^{D-1}\,\mathrm{d}r \tag{3.5}$$

which is the same integral as Stillinger's in equation (2.4).

3.2. Multi-variable integration

A shortfall of the integral in equation (3.5) is that it is only over a single variable, and not multiple variables as in equation (2.1), and is only useful for integrating spherically symmetric functions. For example only the single-coordinate Laplacian can be constructed from equation (3.5). In what follows we will extend the measure theory method to multiple variables by considering product spaces and product measures.

We begin with a collection of n measure spaces (X_m, M_m, μ_m, d) equipped with a metric d and m, a positive integer, labels a specific measure space. We take $X \subset \mathbb{R}^1$ with elements of X denoted by 1-tuples of real numbers $x = (x^m)$. We then form a Cartesian product of all the sets X_n producing the space $X_1 \times X_2 \times \cdots \times X_n$.

The definition of product measures and application of Fubini's theorem (appendix A) provides us with a measure of the product space $X_1 \times X_2 \times \cdots \times X_n$ as

$$(\mu_1 \times \cdots \times \mu_n)(X_1 \times \cdots \times X_n) = \mu_1(X_1)\cdots\mu_n(X_n). \tag{3.6}$$

We note that we are using a product measure here which does not in general need to be a Hausdorff measure [29]. Integration over a function f on the product space can be written as

$$\int f(x^1,\ldots,x^n)\,\mathrm{d}(\mu_1 \times \cdots \times \mu_n) = \int \cdots \int f(x^1,\ldots,x^n)\,\mathrm{d}\mu_1(x^1)\cdots\mathrm{d}\mu_n(x^n). \tag{3.7}$$

In this form, the single-variable measure from equation (3.5) may now be used for each coordinate, which in turn has an associated dimension α_n,

$$\mathrm{d}\mu_n(x^n) = W_1(x^n,\alpha_n)\,\mathrm{d}x^n$$
$$= \sigma(\alpha_n)(x^n)^{\alpha_n-1}\,\mathrm{d}x^n \tag{3.8}$$

and the total dimension of the product space is set as $D := \sum_i \alpha_i$.

As a first example of the approach, we consider a two-variable problem specified by the product space $X_1 \times X_2$ with measures α_1 and α_2. Setting $\alpha_1 = 1$ so that $\alpha_2 = D - 1$, we find integration in this space as

$$\int \mathrm{d}\mu(x^1)\,\mathrm{d}\mu(x^2) f(x^1,x^2) = \int \mathrm{d}x^1\,\mathrm{d}x^2 \sigma(D-1)(x^2)^{D-2} f(x^1,x^2) \tag{3.9}$$

which is essentially the same as Stillinger's two-variable integration expression in equation (2.8), and in turn leads to the two-variable Laplacian in equation (2.9). In essence we have formed a product space comprising X_2 with a Hausdorff measure of dimension $D-1$ and X_1 with the Lebesgue measure. The construction is reflected in the form of the Laplacian, equation (2.9), with one coordinate unchanged from the usual Laplacian while the other has an additional first-order derivative term.

For our second example we look to reproduce the result for the single-variable integration, equation (3.5), from the product space $X_1 \times X_2 \times \cdots \times X_n$. This result is used later to show that Stillinger's approach [6] is contained in the product space method. We take a spherically

symmetric function $f(x^1, \ldots, x^n) \to f(r)$ where $r^2 = (x^1)^2 + \cdots + (x^n)^2$ and perform the integration in spherical coordinates $(r, \theta_1, \ldots, \theta_{n-1})$. Equation (3.7) becomes

$$\int \mathrm{d}\mu_1(x^1) \cdots \mathrm{d}\mu_n(x^n) f(r) = \sigma(\alpha_1) \cdots \sigma(\alpha_n) \int \mathrm{d}r \int \mathrm{d}\theta_1 \cdots \int \mathrm{d}\theta_{n-1} J_n r^{\alpha_1+\cdots+\alpha_n-n} \times (\cos\theta_1)^{\alpha_1-1} (\sin\theta_1)^{\alpha_2+\cdots+\alpha_n-(n-1)} (\cos\theta_2)^{\alpha_2-1} \times (\sin\theta_2)^{\alpha_3+\cdots+\alpha_n-(n-2)} \cdots (\sin\theta_{n-1})^{\alpha_n-1} f(r) \tag{3.10}$$

where J_n is the Jacobian of the coordinate change given by [10]

$$J_n = r^{n-1} (\sin\theta_1)^{n-2} (\sin\theta_2)^{n-3} \cdots (\sin\theta_{n-2}) \times 1.$$

Since the function being integrated is only dependent on the radial variable and not the angular variables we can integrate these using the standard integral [30]

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \, \mathrm{d}x = \frac{1}{2} \frac{\Gamma(\mu/2)\Gamma(\nu/2)}{\Gamma\left(\frac{\mu+\nu}{2}\right)}$$

providing $\mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$. With $D = \sum_{i=1}^n \alpha_i$ we obtain

$$\int \mathrm{d}\mu_1(x^1) \cdots \mathrm{d}\mu_n(x^n) f(r) = \sigma(D) \int f(r) r^{D-1} \, \mathrm{d}r \tag{3.11}$$

which describes integration over a spherically symmetric function in a D-dimensional space and reproduces the earlier result in equation (3.5).

The relationship established in equation (3.11) can be used to show that Stillinger's approach is contained in the product space method. In Stillinger's formulation the integration weights in equation (2.1) are defined as functions over the $\frac{1}{2}n(n+1)$ distances r_{ij} whereas the weights are a function of n variables. This can be seen by considering the relation between weights of equation (2.3). For example we can write W_1 in terms of W_n as

$$W_1(\mathbf{x}_1 | r_{01}) = \int_0^\infty \cdots \int_0^\infty \mathrm{d}r_{0n} \cdots \mathrm{d}r_{02} W_n(\mathbf{x}_1 \cdots \mathbf{x}_n | r_{01} \cdots r_{0n}) \tag{3.12}$$

where W_1 is a function of one variable. In the $n-1$ integrals above we are essentially integrating out a variable in each of these, leaving us with a function of a single variable. It follows that the W_n must be functions of n variables and not $\frac{1}{2}n(n+1)$ variables. In this respect [6] appears to be using a coordinate system which introduces extra variables which are not necessary.

A connection between the two approaches is evident when calculating the volume of a D-dimensional ball. We begin with the characteristic function over which the integration takes place, $\chi(r, R)$, defined by

$$\chi(r, R) = \begin{cases} 1 & \text{if} \quad 0 \leqslant r \leqslant R \\ 0 & \text{if} \quad r > R. \end{cases}$$

With the help of equation (3.12) the volume of a D-dimensional ball, from Stillinger's approach, can be written as

$$\begin{aligned} V_D(R) &= \int_0^\infty W_1(\mathbf{x}_1 | r_{01}) \chi(r_{01}, R) \\ &= \int_0^\infty \mathrm{d}r_{0n} \cdots \int_0^\infty \mathrm{d}r_{01} W_n(\mathbf{x}_1 \cdots \mathbf{x}_n | r_{01} \cdots r_{0n}) \chi(r_{01}, R) \\ &= \int \mathrm{d}x^1 \cdots \int \mathrm{d}x^n J_n' W_n'(x^1 \cdots x^n) \chi(r_{01} = g_1(x^1, \ldots, x^n), R) \end{aligned} \tag{3.13}$$

where for the final step we have performed some coordinate transformation $r_{0n} = g_n(x^1, \ldots, x^n)$ with J'_n being the Jacobian of this transformation.

We can write a similar expression for the volume of a D-dimensional ball from the point of view of Hausdorff measures in a product space. Since $\chi(r, R)$ is a function of a single variable we interpret it as a radial function and, following equation (3.11), integrate over a product space

$$V_D(R) = \int \mathrm{d}\mu_1(x^1) \ldots \mathrm{d}\mu_n(x^n)\chi(r, R). \tag{3.14}$$

Both equations (3.13) and (3.14) are calculating the same quantity by integrating over the same function. If we were to choose the coordinate transformation in equation (3.13) so that it is in spherical or Cartesian form we could equate the integration weights with the measures above. In general some care must be exercised since the weights (W_n) in equation (3.13) depend only on one free parameter whereas for the measures in equation (3.14) we have n free parameters. For integrating over a spherically symmetric function this does not matter, as we have seen above, but otherwise it is important. To equate the weights leaving one free parameter in $\mathrm{d}\mu_1(x^1) \cdots \mathrm{d}\mu_n(x^n)$ we may choose either $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$ and the free parameter $\alpha_n = D - (n-1)$ or $\alpha_1 = \alpha_2 = \cdots = \alpha_n = D/n$. It is not clear in general which case should be used although we expect the former because this is what occurred for integration over two variables in equation (2.8). In addition, for the latter case the system is isotropic and can therefore be rejected for a description of anisotropy. Extending this to n variables leaves one of the sets in the product space with the Hausdorff dimension of $D - (n-1)$ with the remaining characterized by Lebesgue measures. In this case the integrand term in the last line of equation (3.13) can be written for an orthogonal system as

$$J'_n(x^1 \cdots x^n)W'_n(x^1 \cdots x^n) \to W_1(x^n, D - (n-1))$$

so that integration in this space is expressed as

$$\int \mathrm{d}\mu(x^1) \cdots \mathrm{d}\mu(x^n) f(x^1 \cdots x^n) = \int \mathrm{d}x^1 \cdots \mathrm{d}x^n \sigma(D - (n-1))(x^n)^{D-n} f(x^1 \cdots x^n). \tag{3.15}$$

This result not only reproduces the earlier results taken from Stillinger's method, i.e. equations (2.4) and (2.8) when $n = 1$ and $n = 2$ respectively, it also extends and generalizes the method in a straightforward manner to n variables. On this later point, we could assign a dimension to each coordinate rather than a single parameter to the whole system. The product space method can therefore be used to address a question brought up in the introduction. When specifying a problem we choose the number of orthogonal coordinates (usually 3) and the dimension associated with the coordinate to represent the degree of confinement in that direction. The total dimension of the system is simply the sum of the individual dimensions.

4. Euler–Lagrange equations in non-integer dimensions

In this section we use the variational principle to derive Euler–Lagrange equations for fields in non-integer dimensions. In the first instance this provides an alternative derivation of Stillinger's Laplacians, although the scope of the approach is far wider. Given an appropriate Lagrange density for a system we wish to describe, this method provides a straightforward approach to obtaining the dynamics of the fields (the equation of motions) for any number of degrees of freedom, and significantly, in any coordinate basis. Our approach closely follows the standard method for obtaining the Euler–Lagrange equations [10, 31, 32] arriving at a general system described by a Lagrangian density in n coordinates. We note that recent

independent work briefly considered a one-coordinate system describing shallow donor s-states [24]. Equations of motion have also been considered in the context of fractional calculus [5, 33] where the derivatives in the Euler–Lagrange equations are replaced by fractional derivatives.

The action principle for a functional L (the Lagrangian), dependent on the field $\phi(x)$ and its spatial and time derivatives $\partial_\mu\phi(x)$, is given by the integral

$$S = \int_a^b \mathrm{d}t\, L(\phi(x), \partial_\mu\phi(x))$$

with integration between initial and final times a and b. The action can be written in terms of a Lagrange density $L = \int_{\partial\Omega} \mathrm{d}^D x \mathcal{L}(\phi(x), \partial_\mu\phi(x))$, where D is the dimension of space and $\partial\Omega$ is the boundary. In terms of the Lagrangian density we have

$$S = \int_{\partial\Omega'} \mathrm{d}^{D+1} x \mathcal{L}$$

where $\partial\Omega'$ is the boundary for all coordinates. To find the equation of motion of a field we look to minimize the action, i.e. $\delta S = 0$. We consider small variations from the exact solution $\phi_0(x)$ and $\partial_\mu\phi_0(x)$ as $\phi(x) = \phi_0(x) + \delta\phi(x)$ and $\partial_\mu\phi(x) = \partial_\mu\phi_0(x) + \delta(\partial_\mu\phi(x))$ with $\frac{\delta\phi(x)}{\phi(x)} \ll 1$. The variation does not occur at the end points or boundaries and gives conditions on the fields such that $\delta\phi(x)|_{\partial\Omega} = 0$ and $\delta(\partial_\mu\phi(x))|_{\partial\Omega} = 0$.

By taking variations and minimizing the action, the usual Euler–Lagrange equations are obtained [10, 31, 32],

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0. \tag{4.1}$$

The extension to non-integer-dimensional space proceeds in a similar manner, but now with each orthogonal spatial coordinate having its own separate measure as in equation (3.8). In what follows we keep time in integer dimensions and concentrate on non-integer spatial dimensions. The action principle for n degrees of freedom is

$$S = \int \mathrm{d}t\, \mathrm{d}^D x \mathcal{L}(\phi, \partial_\mu\phi) = \int \mathrm{d}t \int \mathrm{d}\mu_1(x^1)\cdots\mathrm{d}\mu_n(x^n)\mathcal{L}(\phi, \partial_\mu\phi) \tag{4.2}$$

where ϕ and $\partial_\mu\phi$ are functions of $(t, x^1, \ldots, x^n)$ and $\partial_\mu = (\partial/\partial t, \partial/\partial x^i)$ with i running from 1 to n. Using the measure in equation (3.8) the integration range extends from zero to infinity. The variation of the action with respect to the fields can be written as

$$\begin{aligned}\delta S &= \int \mathrm{d}t \int \mathrm{d}\mu_1(x^1)\cdots\mathrm{d}\mu_n(x^n)\left[\frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right] \\ &= \int \mathrm{d}t \int \mathrm{d}x^1\cdots\mathrm{d}x^n \prod_{j=1}^n W_1(x^j,\alpha_j)\left[\frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right]\end{aligned} \tag{4.3}$$

where in the last step we have used equation (3.8) for the measures. After integrating the second term in equation (4.3) by parts the total variation becomes

$$\begin{aligned}\delta S = \int \mathrm{d}t \int \mathrm{d}x^1\cdots\mathrm{d}x^n\, \delta\phi\Bigg[&\prod_{j=1}^n W_1(x^j,\alpha_j)\frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial\phi} \\ &- \prod_{j=1}^n W_1(x^j,\alpha_j)\partial_\mu\frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_\mu\phi)}\partial_\mu\prod_{j=1}^n W_1(x^j,\alpha_j)\Bigg].\end{aligned} \tag{4.4}$$

For the extremum in the variation we set $\delta S = 0$; it follows that as this is true for any $\delta\phi$ we arrive at

$$\prod_{j=1}^n W_1(x^j,\alpha_j)\frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial\phi} - \partial_\mu\left[\prod_{j=1}^n W_1(x^j,\alpha_j)\frac{\partial\mathcal{L}(\phi,\partial_\mu\phi)}{\partial(\partial_\mu\phi)}\right] = 0. \tag{4.5}$$

Equation (4.5) is the Euler–Lagrange equation in non-integer dimensions prescribed by measures which take the general form given in equation (3.8). The introduction and influence of the measure distribution will be examined shortly. It is however worth noting that equation (4.5) is generally valid for other types of measures. A straightforward example would be any measures which can be written as $\mathrm{d}\mu(x^j) = \mathcal{W}(x^j)\,\mathrm{d}x^j$; for this situation all instances of W_1 in equation (4.5) would be replaced with $\mathcal{W}$. A more natural requirement of equation (4.5) is that the standard Euler–Lagrange equations should be readily obtained for integer dimensions. This is indeed the case, since for $D = n$ and $\alpha_1 = 1, \ldots, \alpha_n = 1$, the measures in equation (4.5) become $\prod_{j=1}^{n} W_1(x^j, \alpha_j) = 1$ in Cartesian coordinates from which $\partial_\mu \prod_{j=1}^{n} W_1(x^j, \alpha_j) = 0$ and the standard Euler–Lagrange equations (4.1) are recovered.

Apart from a simple scaling, the introduction of non-integer dimensions results in an additional term not present for integer dimensions, i.e. the third term in the integrand of (4.4). In this form, the extra term comprises a 'flow' or 'current' of the measure $\partial_\mu \prod_{j=1}^{n} W_1(x^j, \alpha_j)$ multiplied by $\frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)}$, the canonical momentum density of the field $\mathcal{L}$ in integer dimensions. We see that in general the dynamics of the field described by $\mathcal{L}$ is altered by the dimension. Indeed the use of describing anisotropy by altering the space is often referred to as 'dynamic space' [3].

This role can be further appreciated by writing equation (4.5) in Cartesian form. An intermediate step requires the evaluation of the second term in (4.5) using the explicit form of the measures given by (3.8). In this case we find

$$\partial_\mu \prod_{j=1}^{n} W_1(x^j, \alpha_j) = \partial_\mu \prod_{j=1}^{n} \sigma(\alpha_j)(x^j)^{\alpha_j - 1}$$
$$= (\alpha_\mu - 1)(x^\mu)^{-1} \prod_{j=1}^{n} W_1(x^j, \alpha_j)$$

where α_μ are the diagonal elements of a matrix which includes both time and spatial dimensions, i.e.

$$\alpha = \mathrm{diag}(1, \alpha_1, \ldots, \alpha_n).$$

Here the total dimension of the system D_t (including time) is given by $D_t = \mathrm{Tr}(\alpha)$ and the spatial dimension D of the system is specified by

$$D = \mathrm{Tr}(\alpha) - 1 \tag{4.6}$$

recalling that for the present work we are keeping time in integer dimensions. With these introductions, the Cartesian form of the Euler–Lagrange equations in non-integer dimensions is written as

$$\frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} - (\alpha_{\mu\nu} - \delta_{\mu\nu})(x^{(-1)})^\nu \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} = 0 \tag{4.7}$$

with $\delta_{\mu\nu}$ a diagonal unit matrix and we have introduced a vector where the components x^i are inverted, i.e. $x^{(-1)} = \mathrm{column}(t^{-1}, (x^1)^{-1}, \ldots, (x^n)^{-1})$. Comparing this with equation (4.1) we see this produces an extra term not present in integer dimensions.

Before we proceed it is worth commenting on alternative ways the equations of motion in non-integer dimensions could be obtained. One approach would be to use individual momentum operators in [13–15] for orthogonal coordinates with a different parameter for each coordinate. The product Hausdorff measure space could then be used to identify the dimension on the momentum operators as in [13] providing another multiple-variable approach to the equations of motion.

An attraction of developing the equations using variational principles is that conserved quantities may be easily identified and offer further insight into the role of the measure distribution. The general approach is usually known as Noethers theorem [10, 32] and, before a specific example is examined in the following section, we investigate the conserved quantities arising from our general (non-integer dimension) equations given by (4.5). Formally we begin in much the same way, but instead of small functional variations from exact solutions, we consider symmetry transformations of the system $\phi(x) \to \phi(x) + \delta\phi(x)$ which will leave the action invariant. The resulting variation of action takes the same form as equation (4.3) which can now be reworked using the Euler–Lagrange equations for non-integer dimensions in (4.5). We arrive at

$$\delta S = \int \mathrm{d}t \int \mathrm{d}x^1 \cdots \mathrm{d}x^n \, \partial_\mu \left[\prod_{j=1}^{n} W_1(x^j, \alpha_j) \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} \delta\phi \right]. \tag{4.8}$$

Under a continuous symmetry transformation the action is invariant and we have

$$\partial_\mu \left[\prod_{j=1}^{n} W_1(x^j, \alpha_j) \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} \delta\phi \right] = 0$$

a result which describes a conservation law for an associated conserved current density, i.e.

$$\partial_\mu J^\mu = 0. \tag{4.9}$$

where

$$J^\mu = \prod_{j=1}^{n} W_1(x^j, \alpha_j) \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial(\partial_\mu \phi)} \delta\phi.$$

In the conventional approach (i.e. integer dimensional) the resulting current is often referred to as a Noether current [10, 32]. In this respect equation (4.9) describes a Noether current in non-integer dimensions where, as with earlier observations, the measure distribution of the space is clearly seen to influence the dynamic quantities. We note the integer dimension result is recovered as $\prod_{j=1}^{n} W_1(x^j, \alpha_j) = 1$.

Finally it is worth noting that the form of the key equations presented above, i.e. (4.5) and (4.9), could just have easily been obtained from the standard integer results using the substituion, $\mathcal{L} \to \prod_{j=1}^{n} W_1(x^j, \alpha_j)\mathcal{L}$ or $J^\mu \to \prod_{j=1}^{n} W_1(x^j, \alpha_j) J^\mu$. This useful observation allows a straightforward reworking of standard expressions from integer to non-integer dimensions. For example, the energy–momentum tensor [10] for looking at spacetime translations $x^\mu \to x^\mu + a^\mu$ in non-integer dimensions would become

$$T^\mu_\nu = \prod_{j=1}^{n} W(x^j, \alpha_j) \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu \right] \tag{4.10}$$

with an associated conservation law given by $\partial_\mu T^\mu_\nu = 0$.

5. Example: Schrödinger equation in non-integer dimensions

To obtain the equations of motion all that is required is a Lagrangian density suitable for the problem at hand and application of equation (4.5). In principle we could choose any well-known example (see e.g. [31]). In this section we specifically consider Schrödinger wave mechanics to continue the connection with previous work [6]. However here we can take the advantage of our method to derive both the form and solutions for problems in three coordinates.

The Lagrangian density used in Schrödinger wave mechanics for a single particle in potential V is given as [31]

$$\mathcal{L} = \mathrm{i}\hbar\psi^{\dagger}\dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi^{\dagger}\nabla\psi - V(x^1, \ldots, x^n)\psi^{\dagger}\psi. \tag{5.1}$$

Using equation (4.5) and performing variations with respect to $\psi^{\dagger}$ and $\partial_\mu\psi^{\dagger}$ we obtain for the Schrödinger equation

$$\mathrm{i}\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{\hbar^2}{2m}(\alpha_{ij} - \delta_{ij})(x^{-1})^j\partial_j\psi + V(x^1, \ldots, x^n)\psi \tag{5.2}$$

where i, j run over $1, \ldots, n$, for the n-coordinate system. Written explicitly in Cartesian form for two-spatial coordinate the equation becomes

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y}\right)\psi + V(x, y)\psi. \tag{5.3}$$

In this general form the Laplacian term is notably different from Stillinger's two-coordinate Laplacian in equation (2.9). Equation (5.3) uses two parameters (α_1 and α_2) to describe the measure distribution of space with each one is acting independently on a coordinate and the total spatial dimension of the system is $D = \alpha_1 + \alpha_2$. Following the discussion in the later part of section 3.2 the connection with Stillinger's result is obtained once we set $\alpha_1 = 1$, the total spatial dimensionality becomes $D = \alpha_2 + 1$ and we obtain

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{D - 2}{y}\frac{\partial}{\partial y}\right)\psi + V(x, y)\psi. \tag{5.4}$$

The Laplacian term agrees with Stillinger's result, i.e. equation (6.20) in [6].

Satisfied that our approach is reproducing existing work, we now turn to a problem which, to our knowledge, has not been considered, namely a three-variable Schrödinger equation in non-integer dimensions. As noted in the introduction solutions to such a problem would be useful, for example in exciton problems it would allow the examination of higher orbital states, exploiting the full range of the three principle quantum numbers in non-integer dimensions. For three spatial coordinates, equation (5.2) becomes

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z}\right)\psi + V(x, y, z)\psi. \tag{5.5}$$

Again, following the discussion in the later part of section 3.2, we choose a single parameter for the non-integer dimension, α_3 i.e. $\alpha_1 = \alpha_2 = 1$ so $D = \alpha_3 + 2$. In this case we obtain

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{D - 3}{z}\frac{\partial}{\partial z}\right)\psi + V(x, y, z)\psi. \tag{5.6}$$

For most applications it is likely that a spherical coordinate system (r, θ, ϕ) will be used and in this system (5.6) becomes

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = &-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{D - 1}{r}\frac{\partial}{\partial r}\right)\psi - \frac{\hbar^2}{2m}\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{D - 2}{\tan\theta}\frac{\partial}{\partial\theta}\right)\psi \\ &- \frac{\hbar^2}{2m}\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2} + \frac{D - 3}{\tan\phi}\frac{\partial}{\partial\phi}\right)\psi + V\psi.\end{aligned} \tag{5.7}$$

Note that instead of directly transforming (5.6) the above result can be more easily obtained using the orthonormal basis $\partial_\mu = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$ with equations (4.5) and (5.1).

For a spherically symmetric time-independent potential, $V(t,r,\theta,\phi) \rightarrow V(r)$, equation (5.7) is separable and the wavefunction is written as

$$\psi(r,\theta,\phi,t) \rightarrow R(r)X(\theta)F(\phi)\exp(-\mathrm{i}Et/\hbar) \tag{5.8}$$

resulting in a separable set of equations

$$\left(\frac{1}{r^{D-1}}\frac{\mathrm{d}}{\mathrm{d}r}\,r^{D-1}\frac{\mathrm{d}}{\mathrm{d}r}\right)R(r) + 2[E-V]R(r) + \frac{k_1}{r^2}R(r) = 0 \tag{5.9}$$

$$\left(\frac{1}{\sin^{D-2}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^{D-2}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)X(\theta) - k_1X(\theta) - \frac{k_2}{\sin^2\theta}X(\theta) = 0 \tag{5.10}$$

$$\left(\frac{1}{\sin^{D-3}\phi}\frac{\mathrm{d}}{\mathrm{d}\phi}\sin^{D-3}\phi\frac{\mathrm{d}}{\mathrm{d}\phi}\right)F(\phi) + k_2F(\phi) = 0 \tag{5.11}$$

where k_1 and k_2 are separation constants. Detailed solutions for angular-dependent equations are described in appendix B. We find the appropriate solution to (5.11) is

$$F_m^D(\phi) = T_m^{(D-4)/2}(\cos\phi) \tag{5.12}$$

where $T_m^\beta(x)$ is a Gegenbauer polynomial [31] (see also appendix B (equation (B.3)) for the definition of $T_m^\beta(x)$). The separation constant takes the value

$$k_2 = m(m+D-3) \qquad m = 0,1,2,\ldots. \tag{5.13}$$

The solution to equation (5.9) is found to be

$$X_{l,m}^D(\cos\theta) = (1-(\cos\theta)^2)^{m/2}T_{l-m}^{(D-3)/2+m}(\cos\theta) \tag{5.14}$$

with the separation constant

$$k_1 = -l(l+D-2) \qquad l = 0,1,2,\ldots \qquad \text{where } m \leqslant l. \tag{5.15}$$

The orthogonality relation for the wavefunctions can be found from the orthogonality of the Gegenbauer polynomials described in equation (B.14). So when $x = \cos\phi, z = \cos\theta, \beta = \frac{D-4}{2}$ and $\lambda = \frac{D-3}{2}$ we have

$$\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^\beta(1-z^2)^\lambda F_m^D(x)F_n^D(x)X_{l,m}^D(z)X_{p,n}^D(z)\,\mathrm{d}x\,\mathrm{d}z$$
$$= \delta_{m,n}\,\delta_{l,p}\frac{2\Gamma(n+2\beta+1)2\Gamma(p+2\lambda+m+1)}{(2n+2\beta+1)\Gamma(n+1)(2p+2\lambda+1)\Gamma(p-m+1)} \tag{5.16}$$

which is consistent with the D-dimensional integration weight. (See for example the angular integration weight of equation (3.10) when $n=3, \alpha_1=\alpha_2=1$ and $\alpha_3 = D-2$.)

The radial equation has the same form as Stillinger's, equation (7.7) in [6] which has been solved for a free particle, harmonic oscillator and Coulomb potential in [3, 6]. These solutions are, for a free particle,

$$R(r) = (kr)^{1-(D/2)}J_{(D/2)+l-1}(kr) \qquad k = (2E)^{1/2} \tag{5.17}$$

where $J_{(D/2)+l-1}(kr)$ is a Bessel function. For a harmonic oscillator $V(r) = \frac{1}{2}Kr^2$ the solution is

$$\begin{aligned} &R(r) = \exp\left(-\tfrac{1}{2}s^2\right)s^l L_n^{(D/2)+l-1}(s^2) \\ &s = K^{1/4}r \qquad n = 0,1,2,\ldots \end{aligned} \tag{5.18}$$

with energy eigenvalues

$$E_n = 2K^{1/2}(n+l/2+D/4). \tag{5.19}$$

For the Coulomb potential $V(r) = -Z/r$, solutions are written in terms of the confluent hypergeometric function $M(a, b, z)$ where

$$R(r) = r^l \exp(-\kappa r) M\left(l + \frac{D-1}{2} - (Z/\kappa), 2l + \mathrm{d} - 1, 2\kappa r\right)$$
$$E = -\frac{1}{2}\kappa^2. \tag{5.20}$$

This is square integrable for only discrete values of κ [3, 6] giving the energy spectrum as

$$E_n = \frac{-Z^2}{2\left(n + \frac{D-3}{2}\right)^2} \tag{5.21}$$

where $n = 1, 2, \ldots$ and $l = 0, 1, \ldots, n-1$.

The solution in equation (5.20) and energy spectrum in (5.21) are those used previously to describe excitons in non-integer-dimensional space [3, 16, 20, 22]. Thus irrespective of whether a two- or three-coordinate description is used, the form of the energy spectrum remains the same. The key difference is for the solutions found above we have three quantum numbers $(n,\ l,\ m)$, which are clearly more suitable for describing physical problems set in three coordinates. Stillinger's solutions are formulated on a plane and consequently only provide two quantum numbers $(n,\ l)$. Thus, while the energy spectrum remains the same, the orbital degeneracy between the two cases will in general be different except for situations when $m = 0$.

Comparing the three-coordinate solutions for integer- and non-integer dimensions we note the orbital degeneracy of the m quantum number is reduced, since we are no longer allowed any negative values of m. Degeneracy, however, is retained in the case of the l quantum number. It is noteworthy that similar observations occur for the two-variable case although now it is the l quantum number that is the subject of the reduced degeneracy.

The radial equation is the same as that in Stillinger's so the energy spectrum remains unchanged. If the potential V was angular dependent then we would see an effect of the angular variables in the spectrum. For example, were fine structure effects to be considered, such as spin–orbit coupling, then the angular properties would start to play a part in the splitting of the energy levels.

A more general observation about the solutions in three coordinates is that the square of the total angular momentum, i.e. k_1, takes non-integer values if the dimension is non-integer. This is similar to the quantum mechanics of two non-interacting anyons [34] in which the square of the angular momentum l^2 is replaced by $(l-\alpha)^2$ where α is the anyon parameter occurring from the statistical interaction (see [34] for details). An important difference is that anyons are for two-dimensional systems (two coordinates) whereas in the present case we have similarities with three coordinates and D dimensions.

Finally we consider conserved quantities offered by the action principle in Schrödinger wave mechanics with an explicit example, being a phase transformation with a constant phase of the field [32]

$$\phi(x) \to \phi'(x) = \mathrm{e}^{-\mathrm{i}a}\phi(x) \simeq \phi(x) - \mathrm{i}a\phi(x)$$
$$\phi^\dagger(x) \to \phi'^\dagger(x) = \mathrm{e}^{\mathrm{i}a}\phi^\dagger(x) \simeq \phi^\dagger(x) + \mathrm{i}a\phi^\dagger(x). \tag{5.22}$$

Following the general result obtained earlier, equation (4.9), for the fields ϕ and $\phi^\dagger$ a conserved current is given by

$$J^\mu = \prod_{j=1}^{n} W(x^j, \alpha_j)\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \delta\phi}{\partial a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\frac{\partial \delta\phi^\dagger}{\partial a}\right] \tag{5.23}$$

which satisfies the conservation law of the form of equation (4.9). Using the Lagrangian density in equation (5.1) we find for $\mu = 0$

$$J^0 = \hbar \prod_{j=1}^{n} W(x^j, \alpha_j)\phi^\dagger(x)\phi(x) = \hbar\rho(x) \tag{5.24}$$

whereas for $\mu = i = 1, 2, \ldots, n$

$$J^i = \frac{\hbar^2}{2mi} \prod_{j=1}^{n} W(x^j, \alpha_j)\,[\phi^\dagger(x)\partial_i\phi(x) - (\partial_i\phi^\dagger(x))\phi(x)] = \hbar j^i(x). \tag{5.25}$$

The quantities $\rho(x)$ and $j^i(x)$ are the local probability and current density obeying the continuity equation (4.9) and reduce to the well-known forms in integer dimensions [10, 31, 32]. We see that the effect of the non-integer dimensions is to modify the probability and current density so that it takes into account the measure distribution of the space. The measure then appears in the continuity equation above because it is crucial in keeping the properties of non-integer-dimensional space in this equation.

6. Conclusion

We have extended the measure theoretic approach of Svozil [7] to a multiple-variable space where each orthogonal coordinate has its own dimension. This extension is a significant generalization of the existing multiple-variable approaches allowing for the description of different degrees of confinement in different directions. The method has been applied with the variational principle to obtain the Euler–Lagrange equations in non-integer dimensions, clearly identifying terms which are not present in integer dimensions. The overall approach allows for a much simpler derivation of the non-integer-dimensional Laplacians but also may be applied to any system in any number of coordinates provided we have an appropriate Lagrange density. As an example of the approach, we solved the time-independent Schrödinger equation in three coordinates under numerous potentials. We observed that there was a reduction in the degeneracy of the energy levels and that the square of the angular momentum takes on non-integer values depending on the dimension.

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Appendix A. Product measures

Here we look briefly at some of the main theorems involved in forming the Cartesian product of two measure spaces. A full description of this is given in [28] containing proofs for the theorems below.

When α measures X and β measures Y, we define the function

$$\alpha \times \beta : \mathbf{2}^{X\times Y} \to \overline{\mathbb{R}} \cap \{t : t \geqslant 0\}$$

so that, for any $S \subset X \times Y$, $(\alpha \times \beta)S$ is the infimum of the numbers

$$\sum_{j=1}^{\infty} \alpha(A_j) \cdot \beta(B_j)$$

corresponding to all sequences of α measurable sets A_j and β measurable sets B_j with

$$S \subset \bigcup_{j=1}^{\infty} A_j \times B_j.$$

$\alpha \times \beta$ measures $X \times Y$, and

$$(\alpha \times \beta)(A \times B) \leqslant \alpha(A) \cdot \beta(B)$$

whenever A is α measurable and B is β measurable. $\alpha \times \beta$ is the largest measure satisfying this inequality and is called the Cartesian product of α and β.

The following theorems we look at are part of Fubini's theorem. Suppose α measures X and β measures Y. Now, if A is an α measurable set and B is a β measurable set, then $A \times B$ is an $\alpha \times \beta$ measurable set and

$$(\alpha \times \beta)(A \times B) = \alpha(A) \cdot \beta(B). \tag{A.1}$$

If f is an $\alpha \times \beta$ integrable and countably $\alpha \times \beta$ measurable function (in particular, if f is $A \times \beta$ summable), then

$$\int f \, \mathrm{d}(\alpha \times \beta) = \iint f(x, y) \, \mathrm{d}\alpha(x) \, \mathrm{d}\beta(y) = \iint f(x, y) \, \mathrm{d}\beta(y) \, \mathrm{d}\alpha(x). \tag{A.2}$$

These results can then easily be extended to products of n spaces.

Appendix B. Solution to angular equations

Here we find the appropriate solutions to equations (5.10) and (5.11). The solutions of which are true for any radial potential. Starting with (5.11) we make the substitution $x = \cos\phi$ to obtain

$$\left((1 - x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (D - 2)\, x \frac{\mathrm{d}}{\mathrm{d}x} + k_2\right) F(x) = 0. \tag{B.1}$$

If we write $\beta = \frac{D-4}{2}$ and set $k_2 = m(m + 2\beta + 1)$ where m is a real number, then we obtain a form of the Gegenbauer equation [31]

$$(1 - x^2)F'' - 2(\beta + 1)xF' + m(m + 2\beta + 1) = 0. \tag{B.2}$$

The solution of which is a Gegenbauer polynomial $T_m^{\beta}(x)$ [31]. If m is zero or a positive integer then the solution is analytic at $x = \pm 1$ and the polynomial is finite [31]. Then we can write the solution to this equation as [31]

$$T_m^{\beta}(x) = \frac{(-1)^m \Gamma(m + 2\beta + 1)}{2^{m+\beta} m! \Gamma(m + \beta + 1)} (1 - x^2)^{-\beta} \frac{\mathrm{d}^m}{\mathrm{d}x^m} (1 - x^2)^{m+\beta}. \tag{B.3}$$

Note that the Gegenbauer polynomial here is slightly different to that in [6, 3] by a multiplicative factor. So the appropriate solution to equation (5.11) is

$$F_m^D(\phi) = T_m^{(D-4)/2}(\cos\phi) \tag{B.4}$$

where $m = 0, 1, 2, \ldots$ with separation constant

$$k_2 = m(m + D - 3). \tag{B.5}$$

We now solve equation (5.10). With k_2 found as above and setting $\lambda = \frac{2\beta+1}{2} = \frac{D-3}{2}$ and with $x = \cos\theta$ again, equation (5.10) becomes

$$(1 - x^2)X'' - 2(\lambda + 1)xX' - k_1 X - \frac{m(m + 2\lambda)}{(1 - x^2)} X = 0. \tag{B.6}$$

Considering the case where $m = 0$ and setting $k_1 = -l(l + 2\lambda + 1)$ we have

$$(1 - x^2)X'' - 2(\lambda + 1)xX' + l(l + 2\lambda + 1)X = 0 \tag{B.7}$$

which is of the form of the Gegenbauer equation [31] the solution of which is the Gegenbauer polynomial $T_l^{\lambda}(x)$ defined as above. For an appropriate solution $l = 0, 1, 2, \ldots$ so that $T_l^{\lambda}(x)$ is analytic at $x = \pm 1$ and is also a finite polynomial. For $m = 0$ the solution is $X_l^D(\theta) = T_l^{(D-3)/2}(\cos\theta)$ and the separation constant is

$$k_1 = -l(l + 2\lambda + 1) = -l(l + D - 2) \tag{B.8}$$

where $l = 0, 1, 2, \ldots$.

For $m \neq 0$, equation (B.6) becomes

$$(1 - x^2)X'' - 2(\lambda + 1)xX' + l(l + 2\lambda + 1)X - \frac{m(m + 2\lambda)}{(1 - x^2)}X = 0. \tag{B.9}$$

We try the following form for the solution of this equation:

$$X(x) = (1 - x^2)^{m/2}u(x). \tag{B.10}$$

Substituting for this gives the following differential equation in $u(x)$:

$$(1 - x^2)u'' - 2(\lambda + m + 1)xu' + [l(l + 2\lambda + 1) - m(m + 2\lambda + 1)]u = 0. \tag{B.11}$$

If we differentiate (B.7) m times we get equation (B.11) where $U = \frac{\mathrm{d}^m}{\mathrm{d}x^m}T_l^{\lambda}(x)$. So the solution of (B.9) is $X_{l,m}^D(x) = (1 - x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}T_l^{\lambda}(x)$. Because T is a polynomial of order l then we have the condition $m \leqslant l$ like in the hydrogen atom, otherwise the wavefunction is zero.

The full solution of equation (5.10) is therefore with $x = \cos\theta$

$$X_{lm}^D(x) = (1 - x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}T_l^{(D-3)/2}(x). \tag{B.12}$$

We note the following relation [31]:

$$\frac{\mathrm{d}}{\mathrm{d}x}T_n^{\beta}(x) = T_{n-1}^{\beta+1}$$

so we can write the solution as

$$X_{l,m}^D(\cos\theta) = (1 - (\cos\theta)^2)^{m/2}T_{l-m}^{(D-3)/2+m}(\cos\theta) \tag{B.13}$$

for $m = 0, 1, 2, \ldots, l = 0, 1, 2, \ldots$ and $m \leqslant l$.

Orthogonality of the Gegenbauer polynomials is found via the following integral [31]:

$$\int_{-1}^{1}(1 - x^2)^{\beta}T_m^{\beta}(x)T_n^{\beta}(x)\,\mathrm{d}x = \delta_{m,n}\frac{2\Gamma(n + 2\beta + 1)}{(2n + 2\beta + 1)\Gamma(n + 1)}. \tag{B.14}$$

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